



TITLE:

Periodic solutions of the short pulse model equation (Mathematical Physics and Application of Nonlinear Wave Phenomena)

AUTHOR(S):

Matsuno, Yoshimasa

CITATION:

Matsuno, Yoshimasa. Periodic solutions of the short pulse model equation (Mathematical Physics and Application of Nonlinear Wave Phenomena). 数理解析研究所講究録 2009, 1645: 114-123

ISSUE DATE:

2009-04

URL:

<http://hdl.handle.net/2433/140670>

RIGHT:

短パルスモデル方程式の周期解 Periodic solutions of the short pulse model equation

山口大学大学院理工学研究科 松野 好雅 (Yoshimasa Matsuno)
Division of Applied Mathematical Science
Graduate School of Science and Engineering
Yamaguchi University

Abstract

The short pulse model equation describes the propagation of ultra-short optical pulses in nonlinear media. We develop a systematic method for solving the short pulse equation and address the construction of the two-phase periodic solutions and their properties. The detail of the content of this paper is described in Ref. [11].

1.1 Maxwell equation

We start from the following Maxwell equation

$$\operatorname{div} \mathbf{D} = \rho, \quad \operatorname{div} \mathbf{B} = 0, \quad \operatorname{rot} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \operatorname{rot} \mathbf{H} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t} \quad (1.1a)$$

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}, \quad \mathbf{B} = \mu_0 \mathbf{H} \quad (1.1b)$$

where \mathbf{E} and \mathbf{H} are electric and magnetic field vectors, respectively, and \mathbf{D} and \mathbf{B} are corresponding electric and magnetic flux density.

We assume that $\rho = 0$ and $\mathbf{j} = 0$ and consider the one-dimensional propagation. Then Eq. (1.1) reduces to

$$\mathbf{E} = E_3(x, t)\mathbf{e}_3, \quad \mathbf{H} = H_2(x, t)\mathbf{e}_2 \quad (1.2)$$

$$\frac{\partial H_2}{\partial x} = \frac{\partial D_3}{\partial t}, \quad \frac{\partial E_3}{\partial x} = \mu_0 \frac{\partial H_2}{\partial t}. \quad (1.3)$$

Using (1.3) and the relation $D_3 = \epsilon_0 E_3 + P_3$, we eliminate H_2 from (1.3) to obtain

$$E_{xx} - \frac{1}{c^2} E_{tt} = P_{tt} \quad (1.4)$$

where we have put $E = E_3$, $P = P_3/(\epsilon_0 c^2)$, $c^2 = (\epsilon_0 \mu_0)^{-1}$. We further assume the relation

$$P = P_{lin} + P_{nl} = \int_{-\infty}^{\infty} \chi(t - \tau) E(x, \tau) d\tau + \chi_3 E^3 \quad (1.5a)$$

$$\chi_{tt} = \chi_0 \delta(t). \quad (1.5b)$$

Substituting (1.5) into (1.4), we obtain the nonlinear wave equation

$$E_{xx} - \frac{1}{c^2} E_{tt} = \chi_0 E + \chi_3 (E^3)_{tt}. \quad (1.6)$$

1.2 Singular perturbation

In accordance with Schäfer and Wayne (2004), we apply the singular perturbation method to Eq. (1.6) to derive the short pulse (SP) equation. We expand E with respect to the small parameter ϵ

$$E(x, t) = \epsilon u_0(\phi, X) + \epsilon^2 u_1(\phi, X) + \dots \quad (1.7a)$$

where the new independent variables ϕ and X are defined by

$$\phi = \frac{t - \frac{x}{c}}{\epsilon}, \quad X = \epsilon x. \quad (1.7b)$$

If we introduce (1.7) into (1.6), we obtain, at the lowest order $O(\epsilon)$, the following PDE

$$-\frac{2}{c} \frac{\partial^2 u_0}{\partial \phi \partial X} = \chi_0 u_0 + \chi_3 \frac{\partial^2 u_0^3}{\partial \phi^2}. \quad (1.8)$$

After an appropriate change of the variables, we arrive at the normalized form of the SP equation:

$$u_{xt} = u + \frac{1}{6} (u^3)_{xx}. \quad (1.9)$$

1.3 Remarks

- The SP equation is a model equation describing the propagation of ultra-short optical pulses in nonlinear media.
- The SP equation has been derived in a mathematical context concerning the integrable PDE (Robelo (1989)).
- The following solutions are known for the SP equation:

Soliton and breather solutions: Sakovich et al (2006), Kuetché et al (2006), Matsuno (2007)

Periodic solutions of traveling type (one-phase solutions): Parkes (2008)

- Analogous integrable equations (Matsuno (2006))

$$u_{xt} = \alpha u + \frac{1}{2} (1 - \beta) u_x^2 - u u_{xx}$$

$\beta = 2$: Short-pulse model for Camassa-Holm equation

$\beta = 3$: Short-pulse model for the Degasperis-Procesi equation, Vakhnenko equation

$\alpha = 0, \beta = 2$: Hunter-Saxton equation

All the above equations have the solutions expressed by the parametric representation.

2. Exact method of solution

2.1 Transformation to the sine-Gordon equation

Introduce the new variable r :

$$r^2 = 1 + u_x^2. \quad (2.1)$$

We rewrite the SP equation (1.9) into the form

$$r_t = \left(\frac{1}{2} u^2 r \right)_x. \quad (2.2)$$

By means of the hodograph transformation $(x, t) \rightarrow (y, \tau)$

$$dy = r dx + \frac{1}{2} u^2 r dt, \quad d\tau = dt \quad (2.3a)$$

or equivalently

$$\frac{\partial}{\partial x} = r \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} + \frac{1}{2} u^2 r \frac{\partial}{\partial y} \quad (2.3b)$$

(2.1) and (2.2) are transformed to

$$r^2 = 1 + r^2 u_y^2, \quad r_\tau = r^2 u u_y. \quad (2.4)$$

Using the transformation

$$u_y = \sin \phi, \quad \phi = \phi(y, \tau) \quad (2.5)$$

(2.4) can be put into the form

$$\frac{1}{r} = \cos \phi. \quad (2.6)$$

It follows from (2.4)-(2.6) that $u = \phi_\tau$. Substituting this into (2.5), we obtain the sine-Gordon(sG) equation :

$$\phi_{y\tau} = \sin \phi. \quad (2.7)$$

We see from (2.3) that $x = x(y, \tau)$ satisfies the following linear PDE

$$x_y = \frac{1}{r}, \quad x_\tau = -\frac{1}{2} u^2. \quad (2.8)$$

2.2 Parametric representation of the solution

Since the integrability of Eq. (2.8), i.e. $x_{y\tau} = x_{\tau y}$ is assured by (2.4), we can integrate (2.8) to obtain

$$x(y, \tau) = \int^y \cos \phi \, dy + d \quad (2.9)$$

where d is an integration constant. The expression of u in terms ϕ is given by

$$u(y, \tau) = \phi_\tau. \quad (2.10)$$

2.3 A criterion for the single-valued functions

To derive a criterion for single-valued functions, we may simply require that $u_x = \tan \phi$ exhibits no singularities. Thus, if

$$-\frac{\pi}{2} < \phi < \frac{\pi}{2}, \quad (\text{mod } \pi), \quad (-\sqrt{2} + 1 < \tan \frac{\phi}{4} < \sqrt{2} - 1). \quad (2.11)$$

then the parametric solutions (2.9) and (2.10) will become single-valued functions for all values of x and t .

3. Periodic solutions

Here, we are concerned with the construction of the periodic solutions of the SP equation, particularly focusing on the two-phase solutions.

3.1 Method of solution

We first introduce the two independent phase variables ξ and η according to

$$\xi = ay + \frac{t}{a} + \xi_0, \quad \eta = ay - \frac{t}{a} + \eta_0 \quad (3.1)$$

where $a(\neq 0)$, ξ_0 and η_0 are arbitrary constants. Then, the sG equation is transformed to

$$\phi_{\xi\xi} - \phi_{\eta\eta} = \sin \phi, \quad \phi = \phi(\xi, \eta). \quad (3.2)$$

We seek solutions of the sG equation of the form

$$\phi = 4 \tan^{-1} \left[\frac{f(\xi)}{g(\eta)} \right]. \quad (3.3)$$

This ϕ satisfies the sG equation provided that

$$f'^2 = -\kappa f^4 + \mu f^2 + \nu \quad (3.4a)$$

$$g'^2 = \kappa g^4 + (\mu - 1)g^2 - \nu. \quad (3.4b)$$

Now, the parametric representation of u follows from (2.10) and (3.3)

$$u = \frac{4}{a} \frac{f'g + fg'}{f^2 + g^2}. \quad (3.5)$$

To obtain the parametric form of x , we note the relation

$$\cos \phi = 1 - \frac{8f^2g^2}{(f^2 + g^2)^2}. \quad (3.6)$$

We modify the right-hand side of (3.6) by introducing the function $Y = Y(\xi, \eta)$

$$Y = \frac{c_1(f^2)' + c_2(g^2)'}{f^2 + g^2}. \quad (3.7)$$

We calculate Y_y . Using (3.4), we can modify this in the form

$$Y_y = \frac{a}{(f^2 + g^2)^2} \left[-2\kappa(c_1 f^6 + 3c_1 f^4 g^2 - 3c_2 f^2 g^4 - c_2 g^6) - 4c_2 f^2 g^2 + 2(c_1 + c_2) \{ -2f g f' g' + 2\mu f^2 g^2 - \nu(f^2 - g^2) \} \right]. \quad (3.8)$$

If we put $c_1 + c_2 = 0$ and $c_1 = -2/a$, then (3.8) simplifies to

$$Y_y = 4\kappa(f^2 + g^2) - \frac{8f^2 g^2}{(f^2 + g^2)^2}. \quad (3.9)$$

If we compare (3.6) and (3.9), we obtain

$$\cos \phi = 1 + Y_y - 4\kappa(f^2 + g^2). \quad (3.10)$$

Finally, substituting (3.10) into (2.9) and integrating, we obtain the parametric representation of x :

$$x = y - \frac{4}{a} \frac{f f' - g g'}{f^2 + g^2} - \frac{4\kappa}{a} \left(\int f^2(\xi) d\xi + \int g^2(\eta) d\eta \right) + d. \quad (3.11)$$

3.2 Examples

Here, we present the three examples of the periodic solutions:

a. Example 1

$$f(\xi) = A \operatorname{cn}(\beta \xi, k_f), \quad g(\eta) = \frac{1}{\operatorname{cn}(\Omega \eta, k_g)} \quad (3.12)$$

$$k_f^2 = \frac{A^2}{1 + A^2} \left(1 + \frac{1}{\beta^2(1 + A^2)} \right) \quad (3.13a)$$

$$k_g^2 = \frac{A^2}{1 + A^2} \left(1 - \frac{1}{\Omega^2(1 + A^2)} \right) \quad (3.13b)$$

$$\Omega^2 = \beta^2 + \frac{1 - A^2}{1 + A^2}. \quad (3.13c)$$

The inequality $0 \leq k_f \leq 1$ implies that the parameter β must be restricted by the condition

$$\frac{A}{\sqrt{1 + A^2}} \leq \beta. \quad (3.14)$$

The parametric solution takes the form

$$u = \frac{4A - \beta \operatorname{sn}(\beta \xi, k_f) \operatorname{dn}(\beta \xi, k_f) \operatorname{cn}(\Omega \eta, k_g) + \Omega \operatorname{cn}(\beta \xi, k_f) \operatorname{sn}(\Omega \eta, k_g) \operatorname{dn}(\Omega \eta, k_g)}{a \frac{A^2 \operatorname{cn}^2(\beta \xi, k_f) \operatorname{cn}^2(\Omega \eta, k_g) + 1}{A^2 \operatorname{cn}^2(\beta \xi, k_f) \operatorname{cn}^2(\Omega \eta, k_g) + 1}} \quad (3.15a)$$

$$x = y + \frac{4\beta}{a} \frac{\operatorname{cn}(\beta \xi, k_f) \operatorname{cn}(\Omega \eta, k_g)}{A^2 \operatorname{cn}^2(\beta \xi, k_f) \operatorname{cn}^2(\Omega \eta, k_g) + 1} \left\{ A^2 \operatorname{sn}(\beta \xi, k_f) \operatorname{dn}(\beta \xi, k_f) \operatorname{cn}(\Omega \eta, k_g) \right.$$

$$\begin{aligned}
& -\frac{\beta k_f^2}{\Omega k_g'^2} \operatorname{cn}(\beta\xi, k_f) \operatorname{sn}(\Omega\eta, k_g) \operatorname{dn}(\Omega\eta, k_g) \Big\} \\
& -\frac{4\beta}{a} \left[E(\beta\xi, k_f) - k_f'^2 \beta\xi - \frac{\beta k_f^2}{A^2 \Omega k_g'^2} \left\{ E(\Omega\eta, k_g) - k_g'^2 \Omega\eta \right\} \right] + d. \quad (3.15b)
\end{aligned}$$

Properties of the solution

- The solution is a multiply periodic function. It becomes a single-valued function if $0 < A < \sqrt{2} - 1$.
- Under the condition $L = m_\xi L_\xi / a = m_\eta L_\eta / a$, $(m_\xi, m_\eta) = 1$ where $L_\xi \equiv 4K(k_f)/\beta$ and $L_\eta \equiv 4K(k_g)/\Omega$, the solution has a period Λ

$$\Lambda = L \left[1 - 4\beta^2 \left\{ \frac{E(k_f)}{K(k_f)} - \frac{k_f^2}{A^2(1-k_g^2)} \frac{E(k_g)}{K(k_g)} + \frac{1}{\beta^2(1+A^2)} \right\} \right] \quad (3.16)$$

where $K(k_f)$ and $E(k_f)$ are the complete elliptic integral of the first and second kinds, respectively. Figure 1 shows a profile of u at $t = 0$ for Example 1.

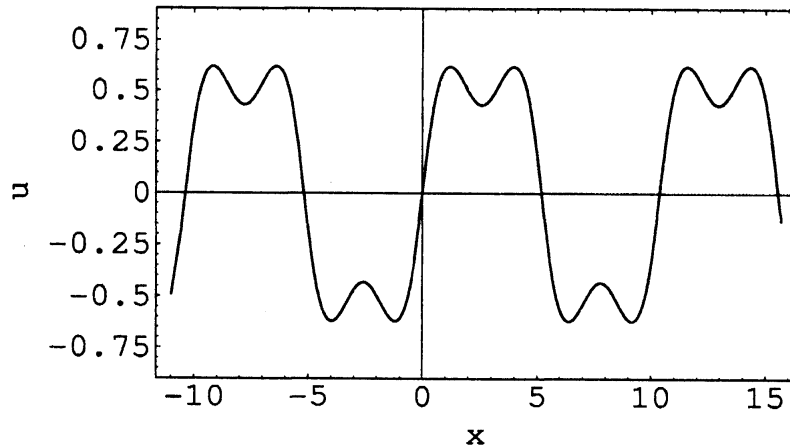


Figure 1: $A = 0.2, m_\xi = 1, m_\eta = 2, a = 1.0, \beta = 0.5832, \Omega = 1.124, k_f = 0.3837, k_g = 0.0958, \Lambda = 10.37$.

Long-wave limit $\Lambda \rightarrow \infty$

In the long-wave limit, the parametric solution reduces to

$$u \sim \frac{4A\Omega}{a} \frac{-A \sinh \beta\xi \cos \Omega\eta + \cosh \beta\xi \sin \Omega\eta}{\cosh^2 \beta\xi + A^2 \cos^2 \Omega\eta} \quad (3.17a)$$

$$x \sim y - \frac{2\Omega \sinh 2\beta\xi + A \sin 2\Omega\eta}{a \cosh^2 \beta\xi + A^2 \cos^2 \Omega\eta} + d. \quad (3.17b)$$

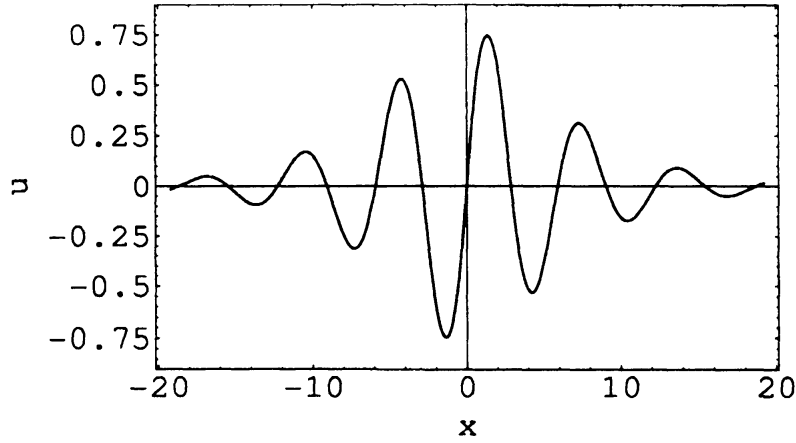


Figure 2: Long-wave limit of the solution depicted in Figure 1.

b. Example 2

$$f(\xi) = A \frac{\text{sn}(\beta\xi, k_f)}{\text{cn}(\beta\xi, k_f)}, \quad g(\eta) = \frac{1}{\text{dn}(\Omega\eta, k_g)} \quad (3.18)$$

$$k_f^2 = 1 - A^2 + \frac{A^2}{\beta^2(1 - A^2)} \quad (3.19a)$$

$$k_g^2 = 1 - \frac{1}{A^2} + \frac{1}{\Omega^2(1 - A^2)} \quad (3.19b)$$

$$\Omega = \beta A \quad (3.19c)$$

$$\frac{1}{\sqrt{1 - A^2}} \leq \beta \leq \frac{1}{1 - A^2} \quad (3.20)$$

$$u = \frac{4A}{a} \frac{\beta \text{dn}(\beta\xi, k_f) \text{dn}(\Omega\eta, k_g) + k_g^2 \Omega \text{sn}(\beta\xi, k_f) \text{cn}(\beta\xi, k_f) \text{sn}(\Omega\eta, k_g) \text{cn}(\Omega\eta, k_g)}{A^2 \text{sn}^2(\beta\xi, k_f) \text{dn}^2(\Omega\eta, k_g) + \text{cn}^2(\beta\xi, k_f)} \quad (3.21a)$$

$$x = y - \frac{4\beta}{a} \frac{1}{A^2 \text{sn}^2(\beta\xi, k_f) \text{dn}^2(\Omega\eta, k_g) + \text{cn}^2(\beta\xi, k_f)} \times$$

$$\times \left[(A^2 \text{dn}^2(\Omega\eta, k_g) - 1) \text{sn}(\beta\xi, k_f) \text{cn}(\beta\xi, k_f) \text{dn}(\beta\xi, k_f) \right.$$

$$\left. + k_g^2 A^3 \text{sn}^2(\beta\xi, k_f) \text{sn}(\Omega\eta, k_g) \text{cn}(\Omega\eta, k_g) \text{dn}(\Omega\eta, k_g) \right] + \frac{4\beta}{a} (-E(\beta\xi, k_f) + AE(\Omega\eta, k_g)) + d \quad (3.21b)$$

$$\Lambda = L \left[1 - 4\beta^2 \left\{ \frac{E(k_f)}{K(k_f)} - A^2 \frac{E(k_g)}{K(k_g)} \right\} \right]. \quad (3.22)$$

Figure 3 shows a profile of u at $t = 5$ for Example 2.

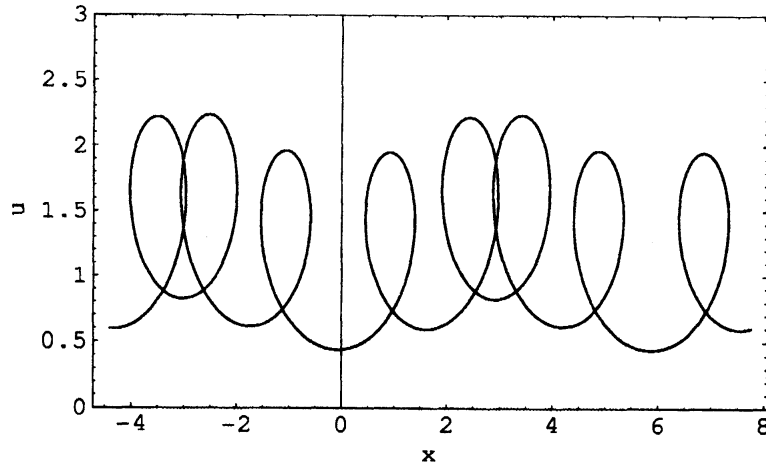


Figure 3: $A = 0.2, m_\xi = 2, m_\eta = 1, a = 1.0, \beta = 1.027, \Omega = 0.2053, k_f = 0.9998, k_g = 0.8421, \Lambda = 5.938$.

Long-wave limit $\Lambda \rightarrow \infty$

$$u \sim \frac{4\beta A \cosh \beta \xi \cosh \Omega \eta + A \sinh \beta \xi \sinh \Omega \eta}{a \frac{A^2 \sinh^2 \beta \xi + \cosh^2 \Omega \eta}{A^2 \sinh^2 \beta \xi + \cosh^2 \Omega \eta}} \quad (3.23a)$$

$$x \sim y - \frac{2\beta A^2 \sinh 2\beta \xi - A \sinh 2\Omega \eta}{a \frac{A^2 \sinh^2 \beta \xi + \cosh^2 \Omega \eta}{A^2 \sinh^2 \beta \xi + \cosh^2 \Omega \eta}} + d. \quad (3.23b)$$

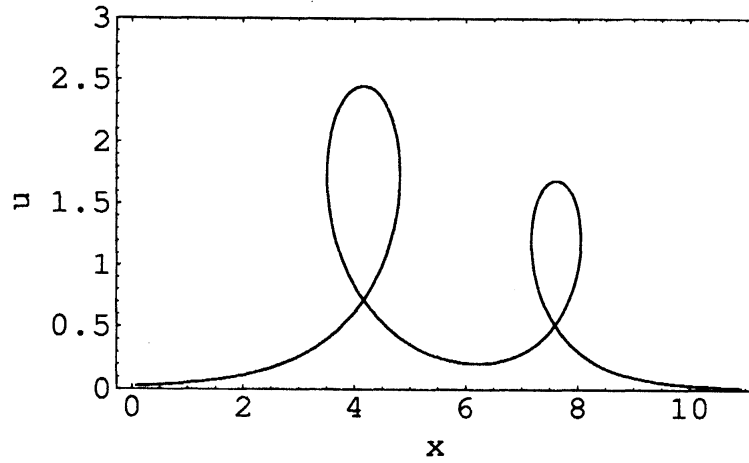


Figure 4: Long-wave limit of the solution depicted in Figure 3.

c. Example 3

$$f(\xi) = A \operatorname{dn}(\beta \xi, k_f), \quad g(\eta) = \frac{\operatorname{cn}(\Omega \eta, k_g)}{\operatorname{sn}(\Omega \eta, k_g)} \quad (3.24)$$

$$k_f^2 = 1 - \frac{1}{A^2} + \frac{1}{\beta^2(A^2 - 1)} \quad (3.25a)$$

$$k_g^2 = 1 - A^2 + \frac{A^2}{\Omega^2(A^2 - 1)} \quad (3.25b)$$

$$\Omega = \frac{\beta}{A} \quad (3.25c)$$

$$\frac{A}{\sqrt{A^2 - 1}} \leq \beta \leq \frac{A^2}{A^2 - 1}, \quad A > 1 \quad (3.26)$$

$$u = -\frac{4A}{a} \frac{\Omega \operatorname{dn}(\beta\xi, k_f) \operatorname{dn}(\Omega\eta, k_g) + \beta k_f^2 \operatorname{sn}(\beta\xi, k_f) \operatorname{cn}(\beta\xi, k_f) \operatorname{sn}(\Omega\eta, k_g) \operatorname{cn}(\Omega\eta, k_g)}{A^2 \operatorname{dn}^2(\beta\xi, k_f) \operatorname{sn}^2(\Omega\eta, k_g) + \operatorname{cn}^2(\Omega\eta, k_g)} \quad (3.27a)$$

$$\begin{aligned} x = y - \frac{4\beta}{a} \frac{1}{A^2 \operatorname{dn}^2(\beta\xi, k_f) \operatorname{sn}^2(\Omega\eta, k_g) + \operatorname{cn}^2(\Omega\eta, k_g)} \times \\ \times \left[\frac{1}{A} (1 - A^2 \operatorname{dn}^2(\beta\xi, k_f)) \operatorname{sn}(\Omega\eta, k_g) \operatorname{cn}(\Omega\eta, k_g) \operatorname{dn}(\Omega\eta, k_g) \right. \\ \left. - k_f^2 A^2 \operatorname{sn}(\beta\xi, k_f) \operatorname{cn}(\beta\xi, k_f) \operatorname{dn}(\beta\xi, k_f) \operatorname{sn}^2(\Omega\eta, k_g) \right] - \frac{4\beta}{a} \left(E(\beta\xi, k_f) - \frac{1}{A} E(\Omega\eta, k_g) \right) + d \end{aligned} \quad (3.27b)$$

$$\Lambda = L \left[1 - 4\beta^2 \left\{ \frac{E(k_f)}{K(k_f)} - \frac{1}{A^2} \frac{E(k_g)}{K(k_g)} \right\} \right]. \quad (3.28)$$

Figure 5 shows a profile of u at $t = 5$ for Example 3.

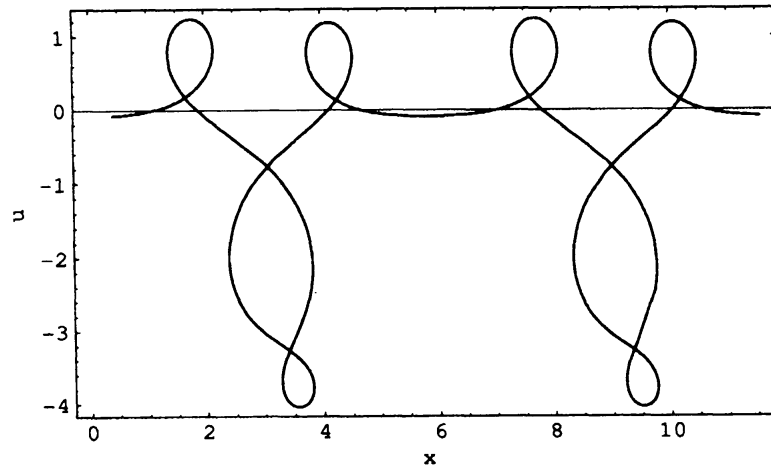


Figure 5: $A = 5, m_\xi = 2, m_\eta = 1, a = 1.0, \beta = 1.027, \Omega = 0.2053, k_f = 0.9998, k_g = 0.8421, \Lambda = 5.938$.

Long-wave limit $\Lambda \rightarrow \infty$

$$u \sim -\frac{4\beta \cosh \beta\xi \cosh \Omega\eta + A \sinh \beta\xi \sinh \Omega\eta}{a \cosh^2 \beta\xi + A^2 \sinh^2 \Omega\eta} \quad (3.29a)$$

$$x \sim y - \frac{2\beta \sinh 2\beta\xi - A \sinh 2\Omega\eta}{a \cosh^2 \beta\xi + A^2 \sinh^2 \Omega\eta} + d. \quad (3.29b)$$

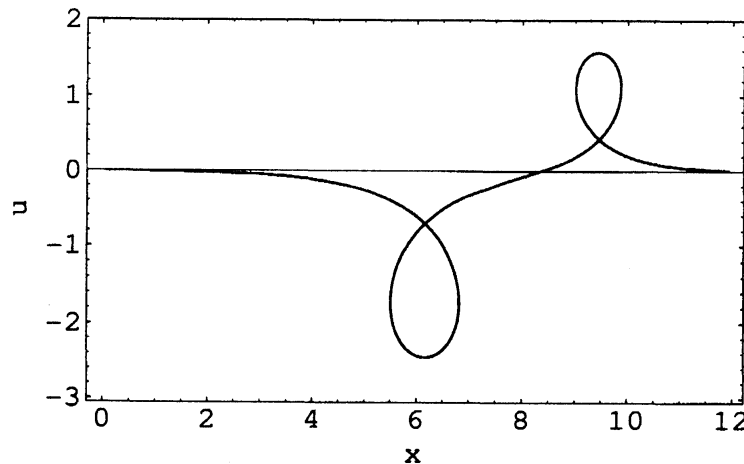


Figure 6: Long-wave limit of the solution depicted in Figure 5.

4. Conclusion

- By means of a novel method of exact solution, we obtained periodic solutions of the SP equation and investigated their properties.
- Of particular interest is the nonsingular periodic solution which reduces to the breather solution in the long-wave limit.
- The construction of a more general class of periodic solutions is under study. It is produced by the multiphase solutions of the sG equation expressed by Riemann's theta functions.

5. References

- [1] T. Schäfer and C. E. Wayne, *Physica D* **196** (2004) 90.
- [2] M. L. Robelo, *Stud. Appl. Math.* **81** (1989) 221.
- [3] R. Beals, M. Rabelo and K. Tenenblat, *Stud. Appl. Math.* **81** (1989) 125.
- [4] A. Sakovich and S. Sakovich, *J. Phys. Soc. Jpn.* **74** (2005) 239.
- [5] J. C. Brunelli, *J. Math. Phys.* **46** (2005) 123507.
- [6] J.C. Brunelli, *Phys. Lett. A* **353** (2006) 475.
- [7] A. Sakovich and S. Sakovich, *J. Phys. A* **39** (2006) L361.
- [8] V. K. Kuetché, T. B. Bouetou and T. C. Kofane, *J. Phys. Soc. Jpn.* **76** (2007) 024004.
- [9] Y. Matsuno, *J. Phys. Soc. Jpn.* **76** (2007) 084003.
- [10] E.J. Parkes, *Chaos, Solitons and Fractals*, **38** (2008) 154.
- [11] Y. Matsuno, *J. Math. Phys.* **49**, 073508 (2008).